MEGA HW1 Solutions
ECNS 561

$= 1 - \binom{12}{0}(.2)^0(1-.2)^{12-0}$
$= 1 – [12!/(0!12!)](.8)^{12}$
$= .931$

(ii)

We want to solve for $P(X \geq 2)$

$P(X \geq 2)$
$= 1 – P(X \leq 1)$
$= 1 – [P(X = 0) + P(X = 1)]$
$= 1 – [.069 + .206]$
$= .725$

B6.

$E(X) = \int_0^3 xf(x)dx$
$= \int_0^3 (x^3/9)dx$
$= \left(\frac{x^4}{36}\right)|_0^3$
$= 2.25$ years

B10. (i)

E(GPA|SAT = 800) = .70 + .002(800) = 2.3

E(GPA|SAT = 1,400) = 3.5

(ii)

Using the law of iterated expectations, we can write

E[E(GPA|SAT)
= E[GPA]
= E(.70 + .002SAT)
= .70 + .002E(SAT)
= .70 + .002(1,100)
= 2.9

2.)

$E(X) = \int_0^1 2x(1-x)dx$
$= (x^2 – 2x^3/3)|_0^1$
$= 1/3$

Var(X) = $\int_0^1 (x - \frac{1}{3})^2 (2 - 2x)dx$
= $2x^3/3 - 4x^2/6 + 2x/9 - x^4/2 + 4x^3/9 - x^2/9$
= $(2x^3/3 - 4x^2/6 + 2x/9 - x^4/2 + 4x^3/9 - x^2/9)_0^1$
= 1/18

3.)

Let X = # of flashlights that work.

Because P(flashlight works) = P(both batteries work), we write

P(both batteries work) = P(B)(B) = (.9)(.9) = .81

We want to solve for $P(X \geq 9)$

$P(X \geq 9) = P(X = 9) + P(X = 10)$
$= \binom{10}{9}(.81)^9(.19) + \binom{10}{10}(.81)^{10}$
= .285 + .122
= . 407

4.)

$E(X) = \sum_{x=1}^{\infty} xp(x) = \sum_{x=1}^{\infty} c/x^2$

A well-known result from the theory of infinite series allows us to write

$\sum_{x=1}^{\infty} c/x^2 < \infty$

So, E(X) is finite.

(In 1735, the famous mathematician Leonhard Euler showed that $\sum_{x=1}^{\infty} 1/x^2 = \pi^2/6$.)

5.)

i.)
$\mu = \sum_{x=0}^{6} xp(x) = 2.64$

$\sigma^2 = [\sum_{x=0}^{6} x^2 p(x)] - \mu^2 = 2.37$. So, $\sigma = 1.54$

For k = 2, we have $\mu - 2\sigma = -.44$ and $\mu + 2\sigma = 5.72$.

So, $P(|X - \mu| \geq k\sigma)$ = P(X is at least 2 s.d.'s from μ)
= P(X is either ≤ -.44 or ≥ 5.72)
= P(X = 6)

= .04.

We see that Chebyshev's upper bound of .25 (= $1/k^2$, where k = 2) is much too conservative.

ii.)

μ = 0 and σ = 1/3

P(|X - μ| ≥ 3σ) = P(|X| ≥ 1)
= P(X= -1 or 1)
= 1/18 + 1/18
= 1/9

This is identical to Chebyshev's upper bound.

iii.)

For μ = 0 and σ = 1/5, we can write

P(|X| ≥ 1)
= P(X= -1 or 1)
= .04

with the corresponding probabilities

P(X = -1) = 1/50, P(X = 0) = 24/25, and P(X = 1) = 1/50

iv.)

P(|X - μ| ≥ σ)
= P(X ≤ μ - σ or X ≥ μ + σ)
= 1 – P(μ - σ ≤ X ≤ μ + σ)
= 1 - P((μ - σ - μ)/σ ≤ Z ≤ (μ + σ - μ)/σ)
= 1 – P(-1 ≤ Z ≤ 1)
= 1 – (Φ(1) – Φ(-1))
= 1 – (.8413 - .1587)
= .3174

6.)

$\int_{-\infty}^{\infty} .15e^{-.15(x-.5)}dx$

$= .15e^{.075} \int_{.5}^{\infty} e^{-.15}dx$

$= .15e^{.075}(\left(\frac{1}{.15}\right) e^{-.075})$ (from the result that $\int_{a}^{\infty} e^{-kx}dx = (1/k)e^{-ka}$)

= 1

7.)

$P(X^2 \leq y) = \int_{-\sqrt{y}}^{\sqrt{y}} (1/\sqrt{2\pi})e^{-\sqrt{x}^2/2}dx$

From the identity given, we can write

$dP(X^2 \leq y)/dy = (1/\sqrt{2\pi})e^{-\sqrt{y}^2/2} \cdot (\frac{1}{2\sqrt{y}})$ - $(1/\sqrt{2\pi})e^{-(-\sqrt{y})^2/2} \cdot (-\frac{1}{2\sqrt{y}})$

$= \frac{e^{-\frac{y}{2}}}{\sqrt{2\pi}\sqrt{y}}$

We know that for all values of $y \geq 0$ and $v = 1$ degrees of freedom, the pdf of a chi-squared random variable is given by

$f(y, 1) = \frac{y^{-1/2}e^{-y/2}}{(2^{1/2} \cdot \Gamma(1/2))}$

$= \frac{e^{-y/2}}{\sqrt{2\pi}\sqrt{y}}$

8.)

i.)

$E[X] = \int_0^\infty xf(x)dx$

$= \int_0^\infty xcx^{\frac{n}{2}-1}e^{-x/2}dx$

$= c\int_0^\infty x^{n/2}e^{-x/2}dx$

$= c\{[-x^{n/2}2e^{-x/2}]|_0^\infty + \int_0^\infty n/2\, x^{\frac{n}{2}-1}2e^{-x/2}dx\}$ (integration by parts)

$= c\{(0\text{-}0) + n\int_0^\infty x^{\frac{n}{2}-1}\, e^{-x/2}dx\}$

$= n\int_0^\infty cx^{\frac{n}{2}-1}\, e^{-x/2}dx$

$= n\int_0^\infty f(x)dx$

$= n$ (recall that the integral of a pdf over its entire support is simply equal to 1)

ii.)

Recall that $Var[X] = E[X^2] - E[X]^2$.

We have already showed that $E[X] = n$. As a result, we know that $E[X]^2 = n^2$.

$E[X^2] = \int_0^\infty x^2f(x)dx$

$= \int_0^\infty x^2cx^{\frac{n}{2}-1}e^{-x/2}dx$

$= c\int_0^\infty x^{\frac{n}{2}+1}e^{-x/2}dx$

$= c\{[-2x^{\frac{n}{2}+1}e^{-\frac{x}{2}}]|_0^\infty + 2\int_0^\infty (\frac{n}{2} + 1)x^{n/2}e^{-x/2}dx\}$ (integration by parts)

$= 2c(n/2 + 1)\int_0^\infty x^{n/2}e^{-x/2}dx$

$= 2c(n/2 + 1)\ \{[-2x^{\frac{n}{2}}e^{-\frac{x}{2}}]|_0^\infty + n\int_0^\infty x^{\frac{n}{2}-1}e^{-x/2}dx\}$ (integration by parts)

$= 2n(n/2 + 1)\{\int_0^\infty cx^{\frac{n}{2}-1}e^{-x/2}dx$

$= 2n(n/2 + 1) \int_0^\infty f(x)dx$
$= n^2 + 2n$

So, we can now write

$\text{Var}[X] = (n^2 + 2n) - n^2$
$= 2n$

9.)

iii.) b.) Real income per capita in 1970 does not appear to be normally distributed. It appears to be skewed toward the upper tail.

iii.) c.) A normal distribution fits the natural log of income better.

iv.) d.) It doesn't appear to have changed a ton over time.