

MEGA HW1 Solutions  
ECNS 561

L)

B2. (i)

$$\begin{aligned} &P(X \leq 6) \\ &= P((X-5)/2 \leq (6-5)/2) \\ &= P((X-5)/2 \leq .5) \\ &= P(Z \leq .5) \\ &= .6915 \end{aligned}$$

(ii)

$$\begin{aligned} &P(X > 4) \\ &= P((X-5)/2 > (4-5)/2) \\ &= P(Z > -.5) \\ &= P(Z \leq .5) \quad (\text{due to symmetry of the standard normal distribution}) \\ &= .6915 \end{aligned}$$

(iii)

$$\begin{aligned} &P(|X-5| > 1) \\ &= P(X-5 > 1) + P(X-5 < -1) \\ &= P(X > 6) + P(X < 4) \\ &= (1-.6915) + (1-.6915) \\ &= .617 \end{aligned}$$

B4. Solve for  $P(X \geq .6)$

$$\begin{aligned} &P(X \geq .6) \\ &= P(X > .6) \quad (\text{because } X \text{ is a continuous random variable}) \\ &= 1 - P(X \leq .6) \\ &= 1 - F(.6) \\ &= 1 - [3(.6)^2 - 2(.6)^3] \\ &= .352 \end{aligned}$$

One way to interpret this is that almost 35.2% of all counties have an elderly employment rate of .6 or higher.

B5. (i)

We want to solve for  $P(X \geq 1)$ .

$$\begin{aligned} &P(X \geq 1) \\ &= 1 - P(X = 0) \end{aligned}$$

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$$\begin{aligned} &= 1 - \binom{12}{0} (.2)^0 (1-.2)^{12-0} \\ &= 1 - [12!/(0!12!)] (.8)^{12} \\ &= .931 \end{aligned}$$

(ii)

We want to solve for  $P(X \geq 2)$

$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 1 - [P(X = 0) + P(X = 1)] \\ &= 1 - [.069 + .206] \\ &= .725 \end{aligned}$$

B6.

$$\begin{aligned} E(X) &= \int_0^3 xf(x)dx \\ &= \int_0^3 (x^3/9)dx \\ &= \left(\frac{x^4}{36}\right) \Big|_0^3 \\ &= 2.25 \text{ years} \end{aligned}$$

B10. (i)

$$E(\text{GPA}|\text{SAT} = 800) = .70 + .002(800) = 2.3$$

$$E(\text{GPA}|\text{SAT} = 1,400) = 3.5$$

(ii)

Using the law of iterated expectations, we can write

$$\begin{aligned} E[E(\text{GPA}|\text{SAT})] &= E[\text{GPA}] \\ &= E(.70 + .002\text{SAT}) \\ &= .70 + .002E(\text{SAT}) \\ &= .70 + .002(1,100) \\ &= 2.9 \end{aligned}$$

2.)

$$\begin{aligned} E(X) &= \int_0^1 2x(1-x)dx \\ &= (x^2 - 2x^3/3) \Big|_0^1 \\ &= 1/3 \end{aligned}$$

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$$\begin{aligned}\text{Var}(X) &= \int_0^1 (x - \frac{1}{3})^2 (2 - 2x) dx \\ &= 2x^3/3 - 4x^2/6 + 2x/9 - x^4/2 + 4x^3/9 - x^2/9 \\ &= (2x^3/3 - 4x^2/6 + 2x/9 - x^4/2 + 4x^3/9 - x^2/9)_0^1 \\ &= 1/18\end{aligned}$$

3.)

Let  $X = \#$  of flashlights that work.

Because  $P(\text{flashlight works}) = P(\text{both batteries work})$ , we write

$$P(\text{both batteries work}) = P(B)(B) = (.9)(.9) = .81$$

We want to solve for  $P(X \geq 9)$

$$\begin{aligned}P(X \geq 9) &= P(X = 9) + P(X = 10) \\ &= \binom{10}{9} (.81)^9 (.19) + \binom{10}{10} (.81)^{10} \\ &= .285 + .122 \\ &= .407\end{aligned}$$

4.)

$$E(X) = \sum_{x=1}^{\infty} xp(x) = \sum_{x=1}^{\infty} c/x^2$$

A well-known result from the theory of infinite series allows us to write

$$\sum_{x=1}^{\infty} c/x^2 < \infty$$

So,  $E(X)$  is finite.

(In 1735, the famous mathematician Leonhard Euler showed that  $\sum_{x=1}^{\infty} 1/x^2 = \pi^2/6$ .)

5.)

i.)

$$\mu = \sum_{x=0}^6 xp(x) = 2.64$$

$$\sigma^2 = [\sum_{x=0}^6 x^2 p(x)] - \mu^2 = 2.37. \text{ So, } \sigma = 1.54$$

For  $k = 2$ , we have  $\mu - 2\sigma = -.44$  and  $\mu + 2\sigma = 5.72$ .

$$\begin{aligned}\text{So, } P(|X - \mu| \geq k\sigma) &= P(X \text{ is at least } 2 \text{ s.d.'s from } \mu) \\ &= P(X \text{ is either } \leq -.44 \text{ or } \geq 5.72) \\ &= P(X = 6)\end{aligned}$$

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$$= .04.$$

We see that Chebyshev's upper bound of .25 ( $= 1/k^2$ , where  $k = 2$ ) is much too conservative.

ii.)

$$\mu = 0 \text{ and } \sigma = 1/3$$

$$\begin{aligned} P(|X - \mu| \geq 3\sigma) &= P(|X| \geq 1) \\ &= P(X = -1 \text{ or } 1) \\ &= 1/18 + 1/18 \\ &= 1/9 \end{aligned}$$

This is identical to Chebyshev's upper bound.

iii.)

For  $\mu = 0$  and  $\sigma = 1/5$ , we can write

$$\begin{aligned} P(|X| \geq 1) \\ &= P(X = -1 \text{ or } 1) \\ &= .04 \end{aligned}$$

with the corresponding probabilities

$$P(X = -1) = 1/50, P(X = 0) = 24/25, \text{ and } P(X = 1) = 1/50$$

iv.)

$$\begin{aligned} P(|X - \mu| \geq \sigma) \\ &= P(X \leq \mu - \sigma \text{ or } X \geq \mu + \sigma) \\ &= 1 - P(\mu - \sigma \leq X \leq \mu + \sigma) \\ &= 1 - P((\mu - \sigma - \mu)/\sigma \leq Z \leq (\mu + \sigma - \mu)/\sigma) \\ &= 1 - P(-1 \leq Z \leq 1) \\ &= 1 - (\Phi(1) - \Phi(-1)) \\ &= 1 - (.8413 - .1587) \\ &= .3174 \end{aligned}$$

6.)

$$\begin{aligned} &\int_{-\infty}^{\infty} .15e^{-.15(x-.5)} dx \\ &= .15e^{.075} \int_{.5}^{\infty} e^{-.15x} dx \\ &= .15e^{.075} \left( \frac{1}{.15} \right) e^{-.075} \quad \left( \text{from the result that } \int_{\alpha}^{\infty} e^{-kx} dx = (1/k)e^{-k\alpha} \right) \\ &= 1 \end{aligned}$$

7.)

$$P(X^2 \leq y) = \int_{-\sqrt{y}}^{\sqrt{y}} (1/\sqrt{2\pi}) e^{-x^2/2} dx$$

From the identity given, we can write

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$$\begin{aligned} dP(X^2 \leq y)/dy &= (1/\sqrt{2\pi})e^{-y^2/2} \cdot \left(\frac{1}{2\sqrt{y}}\right) - (1/\sqrt{2\pi})e^{-(\sqrt{y})^2/2} \cdot \left(-\frac{1}{2\sqrt{y}}\right) \\ &= \frac{e^{-y/2}}{\sqrt{2\pi}\sqrt{y}} \end{aligned}$$

We know that for all values of  $y \geq 0$  and  $v = 1$  degrees of freedom, the pdf of a chi-squared random variable is given by

$$\begin{aligned} f(y, 1) &= \frac{y^{-1/2}e^{-y/2}}{(2^{1/2} \cdot \Gamma(1/2))} \\ &= \frac{e^{-y/2}}{\sqrt{2\pi}\sqrt{y}} \end{aligned}$$

8.)

i.)

$$\begin{aligned} E[X] &= \int_0^\infty xf(x)dx \\ &= \int_0^\infty xc x^{\frac{n}{2}-1} e^{-x/2} dx \\ &= c \int_0^\infty x^{n/2} e^{-x/2} dx \\ &= c \{ [-x^{n/2} 2e^{-x/2}]_0^\infty + \int_0^\infty n/2 x^{\frac{n}{2}-1} 2e^{-x/2} dx \} \quad (\text{integration by parts}) \\ &= c \{ (0-0) + n \int_0^\infty x^{\frac{n}{2}-1} e^{-x/2} dx \} \\ &= n \int_0^\infty c x^{\frac{n}{2}-1} e^{-x/2} dx \\ &= n \int_0^\infty f(x) dx \\ &= n \quad (\text{recall that the integral of a pdf over its entire support is simply equal to 1}) \end{aligned}$$

ii.)

Recall that  $\text{Var}[X] = E[X^2] - E[X]^2$ .

We have already showed that  $E[X] = n$ . As a result, we know that  $E[X]^2 = n^2$ .

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2 f(x) dx \\ &= \int_0^\infty x^2 c x^{\frac{n}{2}-1} e^{-x/2} dx \\ &= c \int_0^\infty x^{\frac{n}{2}+1} e^{-x/2} dx \\ &= c \{ [-2x^{\frac{n}{2}+1} e^{-x/2}]_0^\infty + 2 \int_0^\infty (\frac{n}{2} + 1) x^{n/2} e^{-x/2} dx \} \quad (\text{integration by parts}) \\ &= 2c(n/2 + 1) \int_0^\infty x^{n/2} e^{-x/2} dx \\ &= 2c(n/2 + 1) \{ [-2x^{\frac{n}{2}} e^{-x/2}]_0^\infty + n \int_0^\infty x^{\frac{n}{2}-1} e^{-x/2} dx \} \quad (\text{integration by parts}) \\ &= 2n(n/2 + 1) \{ \int_0^\infty c x^{\frac{n}{2}-1} e^{-x/2} dx \} \end{aligned}$$

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$$\begin{aligned} &= 2n(n/2 + 1) \int_0^{\infty} f(x) dx \\ &= n^2 + 2n \end{aligned}$$

So, we can now write

$$\begin{aligned} \text{Var}[X] &= (n^2 + 2n) - n^2 \\ &= 2n \end{aligned}$$

9.)

iii.) b.) Real income per capita in 1970 does not appear to be normally distributed. It appears to be skewed toward the upper tail.

iii.) c.) A normal distribution fits the natural log of income better.

iv.) d.) It doesn't appear to have changed a ton over time.